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Journal of Geometry and Physics 21 (1997) 279–306

JOURNAL OF GEOMETRY AND PHYSICS

Isotropic tangency

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Received 9 February 1996; revised 6 May 1996

Abstract

This paper proves existence and uniqueness of geodesics through singular points on a real analytic surface M with a real analytic symmetric two-tensor g when the tangent space to the singular set is isotropic at the singular point. This is achieved by introducing a new differentiable structure at the singular point and by introducing the concept semi-analytic curves. In this new differentiable structure the geodesic through the noncritical singular point is up to reparameterization of a semi-analytic curve. This makes it possible to prove existence and uniqueness of geodesics through the singular point. The setting in higher dimensions is indicated.

Using the same techniques we prove the existence of three collision orbits for the Hamiltonian vector field representing Coulomb forces of the Helium atom.

Subj. Class.: Differential geometry
1991 MSC: 53B30, 53B25
Keywords: Geodesic; Degenerate metric; Real analytic vector field

1. Introduction

In this paper we shall be concerned with a real analytic n-dimensional manifold M with a real analytic symmetric two-tensor g. $p \in M$ is a singular point for g provided $g(p)$ is degenerate. Such a point is noncritical provided

$$\mathrm{d}f_p \neq 0, \quad f = \det g_{ij}$$

in some and hence any chart around p. There is an existence and uniqueness theorem for geodesics through p when the tangent space to the set of singular points is nondegenerate, see [3]. The notions strongly radial vector and weakly radial vector are important in this context because geodesics through p have to be tangent to either a strongly radial vector or a weakly radial vector, see [2]. So geodesics can only go across the set of singular points in certain directions. An existence and uniqueness theorem for geodesics tangent to a strongly

PII S0393-0440(96)00021-6

radial vector is proven in [3] while the weakly radial case is handled in [5]. In the weakly radial case the geodesics are smooth curves through the set of singular points in the usual differentiable structure. In the strongly radial case the geodesics can be reparameterized to smooth curves in a new differentiable structure. In both cases one can also prove existence and uniqueness of Jacobi vector fields tangent to the set of singular points.

Parallel transport along smooth curves through the singular set was considered in [1]. Another type of degeneracy is the case where the two-tensor has constant index and coindex, being everywhere degenerate. This appears naturally in Lorentzian manifolds having a lower nonnegative bound on the time-like sectional curvatures. For instance the boundary of the past of a time-like geodesic is a degenerate hypersurface of the Lorentzian manifold of constant index and coindex being everywhere degenerate, see [6].

The isotropic subspace at a singular point p is the subspace

$$I(p) = T_pM \cap T_pM^{\perp}.$$

If the dimension of this subspace is greater than or equal to two then the singular point cannot be noncritical. Despite this fact one can still prove existence and uniqueness of geodesics, see [7].

It is possible that the dimension of the isotropic subspace is one while $\mathrm{d}f_p = 0$. Also in this case there is an existence and uniqueness theorem for geodesics through p, see [4]. A nondegeneracy condition is needed involving the second derivatives of f.

In the present paper we shall focus attention on the case where the singular point is noncritical and the tangent space to the set of singular points is degenerate. In this case the isotropic subspace is a one-dimensional subspace of the tangent space to the set of singular points. So we shall agree to say there is isotropic tangency. To handle the geodesics in this case one has to introduce the concept of semi-analytic curves.

A semi-analytic curve is a curve

$$\gamma(t) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \gamma^{p_1,\ldots,p_k} t^{p_1\alpha_1+\cdots+p_k\alpha_k}, \quad \alpha_i \geq 1, \quad t \in [0, \delta[,$$

$$|\gamma^{p_1,\ldots,p_k}| \leq \frac{C}{(p_1+1)^2 \cdots (p_k+1)^2 r^{p_1+\ldots+p_k}}$$

for some $C, r > 0$, $\delta \in]0, r[$.

A suitable choice of $\gamma^{p_1,\ldots,p_k}$ will make γ an integral curve of a real analytic vector field $A : U \to \mathbb{R}^n$ defined on an open neighbourhood of the origin in $\mathbb{R}^n$, where the linearization of A at 0 has eigenvalues $0 < \lambda_1 < \cdots < \lambda_k$, $\lambda_{k+1}, \ldots, \lambda_n \geq 0$ and $\alpha_i = \lambda_i/\lambda_1$, γ is a unique subject to a choice of

$$\gamma^{1,\ldots,0} = (v_1, 0, \ldots, 0),$$
$$\vdots$$
$$\gamma^{0,\ldots,1} = (0, \ldots, v_k, 0, \ldots, 0).$$

In a new differentiable structure at p there are geodesics through p that can be reparameterized to a semi-analytic curve. The k above depends on the value of a metric invariant defined in terms of the derivatives of g_{ij} at p. The uniqueness of γ above provides an existence and uniqueness theorem for geodesics through p on a real analytic surface M, see the Main Theorem (Theorem 2.4).

In Section 4 we use essentially the same techniques to prove existence of collide orbits for the Hamiltonian vector field representing the Coulomb forces of the Helium atom. These three collision orbits are up to reparametrization semi-analytic curves. The reparametrization function is the inverse of a semi-analytic function. For a recent paper dealing with the Helium atom, see for instance [8].

2. Existence and uniqueness of geodesics

Let (M, g) denote a real analytic n-dimensional manifold with a real analytic symmetric two-tensor g. A singular point p for g is a point where g_p is degenerate and the set of singular points of g is denoted Ξ.

A singular point p for g is noncritical provided

$$\mathrm{d}f_p \neq 0, \quad f = det g_{ij}$$

in some and hence any chart around p. Let us assume p is a noncritical singular point and $T_p\Xi$ is a degenerate subspace of T_pM.

Fix a basis $v_1, \dots, v_n$ in T_pM such that v_1 is isotropic

$$v_1 \in T_pM \cap T_pM^{\perp} = I(p)$$

and $v_2, \dots, v_{n-1} \in T_p\Xi$. We assume that $v_2, \dots, v_n$ are orthogonal. Then

$$v_1 \in T_p\Xi, \quad I(p) = \text{ span } \{v_1\}$$

A chart (U, ϕ) around p is adapted to Ξ and $v_1, \dots, v_n$ provided

$$U \cap \Xi = \{q \in U | \phi_n(q) = 0\}, \quad \phi(p) = 0 \text{ and } \partial_i(p) = v_i,$$

and 0 is a regular value for the restriction of f to U. In such a chart define

$$\mu_k = \frac{\partial g_{1k}}{\partial x_1}(p), \quad \eta = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x_1^2}(p), \quad \alpha_k = \frac{\partial g_{11}}{\partial x_k}(p), \quad \alpha_n = \alpha.$$

Notice that $\alpha \neq 0$ since $\mathrm{d}f_p \neq 0$.

Lemma 2.1. $\eta g_{22} \cdots g_{nn} = \sum_{m=2}^{n} \mu_m^2 g_{22} \cdots \hat{g_{mm}} \cdots g_{nn}$.

In a chart (U, ϕ) adapted to Ξ and $v_1, \dots, v_n$ let X^{ϕ} denote the local representative of the geodesic spray X and define $Y^{\phi} = f X^{\phi}$, where Y has a real analytic extension to TU.

Also define

$$G : \mathbb{R}^n \backslash \{v_1 = 0\} \to \mathbb{R}^n \backslash \{u_1 = 0\}$$
$$v = (v_1, \dots, v_n) \mapsto (v_1, v_1 v_2, \dots, v_1 v_n),$$
$$H(v) = (v_1, v_1 v_2, \dots, v_1 v_n), \quad v \in \mathbb{R}^n$$

with

$$G^{-1}: \mathbb{R}^n \backslash \{u_1 = 0\} \to \mathbb{R}^n \backslash \{v_1 = 0\}$$
$$(u_1, \dots, u_n) \mapsto (u_1, \dots, u_n/u_1).$$

Assume $q > -1$ and define

$$\Phi : \mathbb{R}^n \backslash \{y_1 \le 0\} \to \mathbb{R}^n \backslash \{x_1 \le 0\}$$
$$(y_1, \dots, y_n) \mapsto y_1^{-q}/(q+1)(1, y_2, \dots, y_n)$$

with

$$\Phi^{-1} : \mathbb{R}^n \backslash \{x_1 \le 0\} \to \mathbb{R}^n \backslash \{y_1 \le 0\}$$
$$(x_1, \dots, x_n) \mapsto ((q+1)x_1)^{-1/q}, x_2/x_1, \dots, x_n/x_1).$$

Now define

$$F(v, y) = (G(v), \Phi(y))$$

and the vector field

$$Z^{\phi}(v, y) = DF^{-1}_{F(v,y)}(Y^{\phi}(F(v, y)))y_1^q(q+1).$$

The standard trick from singularity theory enables us to write

$$f(u) = u_n h(u)$$

for a real analytic function h. We find

$$Z_1^{\phi}(v, y) = h \circ G(v) \begin{pmatrix} v_1 v_n \\ v_n(y_2 - v_2) \\ \vdots \\ v_n(y_n - v_n) \end{pmatrix}.$$

In terms of the associated Christoffel symbols $\tilde{\Gamma}_{ij}^k = f\Gamma_{ij}^k$, we find

$$Z_2^{\phi}(v, y) = \begin{pmatrix} -\frac{1}{q} y_1(-\tilde{\Gamma}_{11}^1 - 2\sum_{m=2}^n \tilde{\Gamma}_{1m}^1 y_m - \sum_{k,m=2}^n \tilde{\Gamma}_{km}^1 y_k y_m) \\ y_2(\tilde{\Gamma}_{11}^1 + 2\sum_{m=2}^n \tilde{\Gamma}_{1m}^1 y_m + \sum_{k,m=2}^n \tilde{\Gamma}_{km}^1 y_k y_m) \\ -\tilde{\Gamma}_{11}^2 - 2\sum_{m=2}^n \tilde{\Gamma}_{1m}^2 y_m - \sum_{k,m=2}^n \tilde{\Gamma}_{km}^2 y_k y_m \\ \vdots \\ y_n(\tilde{\Gamma}_{11}^1 + 2\sum_{m=2}^n \tilde{\Gamma}_{1m}^1 y_m + \sum_{k,m=2}^n \tilde{\Gamma}_{km}^1 y_k y_m) \\ -\tilde{\Gamma}_{11}^n - 2\sum_{m=2}^n \tilde{\Gamma}_{1m}^n y_m - \sum_{k,m=2}^n \tilde{\Gamma}_{km}^n y_k y_m \end{pmatrix}.$$

So Z^{ϕ} has a real analytic extension also denoted Z^{ϕ} to $H^{-1}(\phi(U)) \times \mathbb{R}^n$.

Observe that

$$Z^{\phi}(0,0)=0, \quad DZ_0^{\phi}=0$$

due to the fact that

$$g_{11}(p)=\cdots=g_{1n}(p)=0,$$
$$\alpha_i=\frac{\partial g_{11}}{\partial x_i}(p)=0, \quad i=1,\ldots,n-1.$$

Now

$$b_1=\frac{\partial \tilde{\Gamma}_{11}^1}{\partial x_1}(0)=\eta g_{22}\cdot\cdot g_{nn}-\sum_{m=2}^{n}\mu_m g_{22}\cdot\cdot \hat{g_{mm}}\cdot\cdot g_{nn}\tfrac{1}{2}(2\mu_m-\alpha_m)$$
$$=\sum_{m=2}^{n}\mu_m g_{22}\cdot\cdot \hat{g_{mm}}\cdot\cdot g_{nn}\tfrac{1}{2}\alpha_m$$

and

$$\tilde{\Gamma}_{1i}^1=g_{22}\cdot\cdot g_{nn}\frac{1}{2}\frac{\partial g_{11}}{\partial x_i}=\begin{cases}0, & i\neq n,\\ \tfrac{1}{2}\lambda=c_1, & i=n,\end{cases}$$
$$\lambda=g_{22}\cdot\cdot g_{nn}\alpha.$$

For $k\geq 2$ we find

$$\tilde{\Gamma}_{11}^k(0)=0, \quad \frac{\partial \tilde{\Gamma}_{11}^k}{\partial x_1}(0)=0$$

and

$$2b_k=\frac{\partial^2 \tilde{\Gamma}_{11}^k}{\partial x_1^2}(0)$$
$$=-\mu_k g_{22}\cdot\cdot \hat{g_{kk}}\cdot\cdot g_{nn}2\eta$$
$$+(2\eta g_{22}\cdot\cdot \hat{g_{kk}}\cdot\cdot g_{nn}-2\mu_m^2 g_{22}\cdot\cdot \hat{g_{kk}}\cdot\cdot \hat{g_{mm}}\cdot\cdot g_{nn})\tfrac{1}{2}(2\mu_k-\alpha_k)$$
$$+2\mu_k\mu_m g_{22}\cdot\cdot \hat{g_{mm}}\cdot\cdot \hat{g_{kk}}\cdot\cdot g_{nn}\tfrac{1}{2}(2\mu_m-\alpha_m).$$

For $k=n$ this amounts to

$$2b_n=-\eta g_{22}\cdot\cdot g_{n-1,n-1}\alpha-\alpha_m\mu_n\mu_m g_{22}\cdot\cdot \hat{g_{mm}}\cdot\cdot g_{n-1,n-1}$$
$$+\alpha\mu_m^2 g_{22}\cdot\cdot \hat{g_{mm}}\cdot\cdot g_{n-1,n-1}.$$

Furthermore

$$c_i^k=\frac{\partial \tilde{\Gamma}_{11}^k}{\partial x_i}(0)=\alpha_i g_{22}\cdot\cdot \hat{g_{kk}}\cdot\cdot g_{nn}\tfrac{1}{2}(2\mu_k-\alpha_k)=0, \quad i\neq n.$$

Finally

$$d_i^k=\frac{\partial \tilde{\Gamma}_{1i}^k}{\partial x_1}(0)=-\mu_k g_{22}\cdot\cdot \hat{g_{kk}}\cdot\cdot g_{nn}\tfrac{1}{2}\alpha_i=0, \quad i\neq n.$$

Define

$$\mathcal{Z}_2(v, y) = \tfrac{1}{2} D^2 Z_0^{\phi}((v, y), (v, y)).$$

Then

$$\mathcal{Z}_2(v, y) = \begin{pmatrix} \lambda v_1 v_n \\ \lambda v_n (y_2 - v_2) \\ \vdots \\ \lambda v_n (y_n - v_n) \\ -\frac{1}{q} y_1(-b_1 v_1 - 2c_1 y_n) \\ -y_2(-b_1 v_1 - 2c_1 y_n) - b_2 v_1^2 - c_n^2 v_1 v_n - 2d_n^2 v_1 y_n \\ \vdots \\ -y_n(-b_1 v_1 - 2c_1 y_n) - b_n v_1^2 - c_n^n v_1 v_n - 2d_n^n v_1 y_n \end{pmatrix}.$$

Define blow up maps using the euclidean norm $\| \ \|$

$$\begin{aligned} \tilde{F} : \mathbb{R}^{2n} \backslash \{0\} &\to E = \{x \in \mathbb{R}^{2n} | \ \| x \| > 1\} \\ x &\mapsto x \frac{\| x \| + 1}{\| x \|} \end{aligned}$$

$$\begin{aligned} \tilde{G} : I = \{x \in \mathbb{R}^{2n} | 0 < \| x \| < 1\} &\to I \\ x &\mapsto x \frac{\| x \| - 1}{\| x \|}. \end{aligned}$$

To find singular points of the blow up of Z^{Φ} define

$$L = \begin{pmatrix} 1 & \cdot & 0 & k_1 \\ 0 & \cdot & 0 & k_2 \\ \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & 0 & k_n \\ 0 & \cdot & 0 & 0 \\ 0 & \cdot & 0 & l_2 \\ \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & 1 & l_{n-1} \\ 0 & \cdot & 0 & 1 \end{pmatrix},$$

where k_1 is a nonzero solution to

$$b_n k_1^2 + k_1(\tfrac{1}{2} c_n^n + 2d_n^n - b_1) - \tfrac{1}{2}\lambda = 0.$$

We need q to have the value

$$q = \frac{2}{\lambda}(b_1 k_1 + 2c_1) \neq 1$$

and then

$$2k_i = l_i = \frac{2}{\lambda(q-1)}\{b_i k_1^2 + k_1(\tfrac{1}{2}c_n^i + 2d_n^i)\},$$
$$i = 2, \dots, n-1, \quad k_n = \tfrac{1}{2}.$$

Now define

$$W(x) = L^{-1} \circ Z^{\phi} \circ L(x) \quad \text{and} \quad W_2(x) = \tfrac{1}{2} D^2 W_0(x, x).$$

Our definition of the l_i and k_i imply that

$$W_2(0, (y_1, 0, \dots, 0, y_n)) = \tfrac{1}{2}\lambda(0, (y_1, 0, \dots, 0, y_n)).$$

So

$$(0, (y_1, 0, \dots, 0, y_n)), \quad y_1^2 + y_n^2 = 1$$

are singular points of $\mathcal{Z}_*$ where

$$\mathcal{Z}_*(x) = \frac{\| x \|}{\| x \| - 1} D\tilde{F}_{\tilde{F}^{-1}(x)}(W(\tilde{F}^{-1}(x))), \quad \| x \| > 1, \quad \tilde{F}^{-1}(x) \in \mathbb{D}(W),$$
$$\mathcal{Z}_*(x) = \frac{\| x \|}{\| x \| - 1} D\tilde{G}_{\tilde{G}^{-1}(x)}(W(\tilde{G}^{-1}(x))), \quad 0 < \| x \| < 1, \quad \tilde{G}^{-1}(x) \in \mathbb{D}(W),$$

which has a real analytic extension to an open neighbourhood of S^{2n-1}. So has

$$\mathcal{Z}_{**}(x) = \frac{\| x \|}{\| x \| - 1} D\tilde{F}_{\tilde{F}^{-1}(x)}(W_2(\tilde{F}^{-1}(x))), \quad \| x \| > 1.$$

The spectrum of

$$D\mathcal{Z}_*(0, (y_1, 0, \dots, 0, y_n)), \quad y_1^2 + y_n^2 = 1, \quad y_1 > 0, \quad y_n > 0$$

is important to determine the geodesics through p.

We find

$$D\mathcal{Z}_{**}(0, (y_1, 0, \dots, 0, y_n))$$
$$= y_n \begin{pmatrix} (2b_n k_1^2 + k_1(\frac{1}{2}c_n^n + 2d_n^n - b_1)) & 0 & \cdot & (\lambda k_1 + c_n^n k_1^2) & 0 & 0 & \cdot & 0 \\ * & -\lambda & \cdot & * & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2}(2b_n k_1 + (\frac{1}{2}c_n^n + 2d_n^n - b_1)) & 0 & \cdot & (\frac{1}{2}c_n^n k_1 - \frac{1}{2}\lambda) & 0 & 0 & \cdot & 0 \\ * & * & \cdot & * & \frac{1}{2}\lambda y_1^2 & 0 & \cdot & \frac{1}{2}\lambda y_1 y_n \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & * & 0 & \frac{1}{2}(q-1)\lambda & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & * & \frac{1}{2}\lambda y_1 y_n & 0 & \cdot & \frac{1}{2}\lambda y_n^2 \end{pmatrix}.$$

From this we see that $\frac{1}{2}(q-1)\lambda y_n$ and $-\lambda y_n$ are eigenvalues of algebraic multiplicity at least $n-2$. 0 and $\frac{1}{2}\lambda y_n$ are eigenvalues of algebraic multiplicity at least 1. The remaining eigenvalues are the eigenvalues of the matrix

$$y_n \begin{pmatrix} 2b_n k_1^2 + k_1(\frac{1}{2}c_n^n + 2d_n^n - b_1) & \lambda k_1 + c_n^n k_1^2 \\ \frac{1}{2}(2b_n k_1 + (\frac{1}{2}c_n^n + 2d_n^n - b_1)) & \frac{1}{2}c_n^n k_1 - \dfrac{\lambda}{2} \end{pmatrix}. \tag{2.1}$$

We shall now specialize to the case $n = 2$ and let

$$k_1 = -1/2a,$$

a is then a nonzero solution to

$$4a^2 g_{22}(p) - (\alpha + 4\mu_2)a + \eta = 0. \qquad (2.2)$$

Define $b = \mu_2/\alpha$. We shall assume that $\alpha g_{22}(p) < 0$.

When $b = 0, -\frac{1}{8}$ there is one real solution $a_- = a_+ = a < 0$. When $b \in]-\frac{1}{8}, 0[\cup]0, +\infty[$ there are two distinct negative solutions $a_- < a_+$. They give rise to

$$q = q_+ = 2 - \mu_2/2g_{22}(p)a_+, \qquad q = q_- = 2 - \mu_2/2g_{22}(p)a_-.$$

The two values of k_1

$$k_1^+ = -1/2a_+, \qquad k_1^- = -1/2a_-$$

give rise to two linear maps L denoted L_+ and L_-.

The trace of (2.1) is $\sigma = \frac{3}{2}\lambda(q - \frac{5}{3})y_2$ and the determinant is $\delta = \frac{1}{4}\lambda^2(-2+2(q-2)^2)y_2^2$.

So the eigenvalues of (2.1) are $(q-1)\lambda y_2$, $\frac{1}{2}(q-3)\lambda y_2$. So for $q \neq \frac{3}{2}, 3, 4$ the eigenvalues of $D\mathcal{Z}_*(0, y)$ are

$$0, \tfrac{1}{2}\lambda y_2, (q-1)\lambda y_2, \tfrac{1}{2}(q-3)\lambda y_2.$$

Let

$$V^\phi((q-1)\lambda y_2), \quad V^\phi(\tfrac{1}{2}(q-3)\lambda y_2), \quad V^\phi(0), \quad V^\phi(\tfrac{1}{2}\lambda)$$

denote the eigenspaces of $D\mathcal{Z}_*(0, y)$ corresponding to the eigenvalues $(q-1)\lambda y_2$, $\frac{1}{2}(q-3)\lambda y_2$, 0 and $\frac{1}{2}\lambda$.

Definition 2.2. A curve $c :]0, \epsilon[\to TM$ is $\pm$ resolvent semi-analytic provided there exists $\delta \in]0, \epsilon[$ such that

$$c(]0, \delta[) \in \tilde{\phi}^{-1}(\mathbb{D}(F^{-1})), \quad \phi_* = \phi_*^\pm = \tilde{F} \circ L_\pm^{-1} \circ F^{-1} \circ \tilde{\phi}$$

and such that $\phi_* \circ c$ is a semi-analytic curve, see Definition 3.1 for some and hence any chart (U, ϕ) adapted to Ξ and v_1, v_2. We also assume

$$\tilde{F}^{-1} \circ \phi_* \circ c(0) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(L_\pm \circ \tilde{F}^{-1} \circ \phi_* \circ c)_1(0) \neq 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(L_\pm \circ \tilde{F}^{-1} \circ \phi_* \circ c)_3(0) > 0.$$

Lemma 2.3. *Definition 2.2 is well-defined independent of the choice of chart adapted to Ξ and v_1, v_2.*

Proof. We have

$$\phi_* \circ c(t) = d(t),$$
$$F^{-1} \circ \tilde{\phi} \circ c(t) = L \circ \tilde{F}^{-1} \circ d(t) = te(t), \quad e_1(0) \neq 0, \quad e_3(0) > 0.$$

Furthermore

$$\tilde{\phi} \circ c(t) = F(te(t)) = \begin{pmatrix} te_1(t) \\ t^2 e_1(t) e_2(t) \\ (te_3(t))^{-q}/(1+q) \\ (te_3(t))^{-q}/(1+q) te_4(t) \end{pmatrix}.$$

Write

$$D(\psi \circ \phi^{-1}) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

and

$$\psi \circ \phi^{-1}(x) = \begin{pmatrix} x_1 H_{11} + x_2 H_{12} \\ x_2 H_{22} \end{pmatrix}.$$

Then

$$(\tilde{\psi} \circ \tilde{\phi}^{-1}) \circ (\tilde{\phi} \circ c)(t)$$
$$= \begin{pmatrix} t(e_1(t) H_{11} + t e_1 e_2(t) H_{12}) \\ t^2 e_1 e_2(t) H_{22} \\ \frac{(te_3(t))^{-q}}{(1+q)} (\Phi_{11} + t e_4 \Phi_{12}) \\ \frac{(te_3(t))^{-q}}{(1+q)} (\Phi_{21} + t e_4 \Phi_{22}) \end{pmatrix} \in \mathbb{D}(F^{-1}), \quad t \in]0, \delta_1[.$$

Now $\psi_* \circ \phi_*^{-1}$ is the restriction of a real analytic map on a neighbourhood of S^3. Define $a = \phi_* \circ c(0)$ and write

$$\psi_* \circ \phi_*^{-1}(x) = B_{i_1 \dots i_s}(x_{i_1} - a_{i_1}) \cdots (x_{i_s} - a_{i_s}),$$

where

$$|B_{i_1 \cdots i_s}| \leq D/\rho^s, \quad D, \rho > 0.$$

Then

$$\psi_* \circ \phi_*^{-1}(\phi_* \circ c)(t) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \sum_{a=0}^{|\mathbf{p}|} B_{i_1 \cdots i_a} \sum_{\mathbf{p}_1 + \cdots + \mathbf{p}_a = \mathbf{p}} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{pa}}.$$

Define

$$\tilde{\gamma}^{\mathbf{p}} = \sum_{a=0}^{|\mathbf{p}|} B_{i_1 \cdots i_a} \sum_{\mathbf{p}_1 + \cdots + \mathbf{p}_a = \mathbf{p}} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a}.$$

Then

$$\psi_* \circ c(t) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \tilde{\gamma}^{\mathbf{p}} t^{\mathbf{pa}}$$

and assuming $CK_2^k/\rho \geq 2$ we estimate

$$
\begin{aligned}
|\tilde{\gamma}^{\mathbf{p}}| &\leq \sum_{a=0}^{|\mathbf{p}|} D/\rho^a \sum_{\mathbf{p}_1+\cdots+\mathbf{p}_a=\mathbf{p}} \frac{C}{(p_1^1+1)^2\cdots(p_k^1+1)^2}\cdots \\
&\quad \times \frac{C}{(p_1^a+1)^2\cdots(p_k^a+1)^2}\frac{1}{r^{p_1+\cdots+p_k}} \\
&\leq \sum_{a=0}^{|\mathbf{p}|} D(C/\rho)^a \frac{K_2^{a-1}}{(p_1+1)^2}\cdots\frac{K_2^{a-1}}{(p_k+1)^2}\frac{1}{r^{p_1+\cdots+p_k}} \\
&\leq D/K_2^k \frac{(CK_2^k/\rho)^{|\mathbf{p}|+1}}{(CK_2^k/\rho)-1}\frac{1}{(p_1+1)^2\cdots(p_k+1)^2 r^{p_1+\cdots p_k}} \\
&\leq DC/\rho \frac{1}{(p_1+1)^2\cdots(p_k+1)^2(\rho r/CK_2^k)^{p_1+\cdots+p_k}}
\end{aligned}
$$

and $\psi_* \circ c$ is a semi-analytic curve. The lemma follows. □

For $b \in]-\frac{1}{8}, 0[$ define

$$
\begin{aligned}
&\lambda_* = \tfrac{1}{2}\lambda y_2, \\
&\alpha_1 = 1, \quad \alpha_2 = 2(q_+ - 1), \quad \alpha_3, = q_+ - 3, \\
&\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)
\end{aligned}
$$

and assume

$$\mathbf{pa} \neq \alpha_i, \quad i = 1, 2, 3, \quad \mathbf{p} \neq e_1, e_2, e_3.$$

For $b \in]-\frac{1}{8}, +\infty[, b \neq 1$ define

$$
\begin{aligned}
&\lambda_* = \tfrac{1}{2}\lambda y_2, \\
&\alpha_1 = 1, \quad \alpha_2 = 2(q_- - 1), \\
&\mathbf{a} = (\alpha_1, \alpha_2)
\end{aligned}
$$

and assume

$$\mathbf{pa} \neq \alpha_i, \quad i = 1, 2, \quad \mathbf{p} \neq e_1, e_2.$$

Now define

$$V^{\phi}(\tfrac{1}{2}\lambda y_2)^+ = \{v \in V^{\phi}(\tfrac{1}{2}\lambda y_2) | \langle v, (0, y)\rangle > 0\}.$$

Given a geodesic $\gamma :]0, \epsilon[\to M$ with

$$\gamma'(t) \in \mathbb{D}(\phi_*), \quad t \in]0, \epsilon[,$$

consider the differential equation

$$\alpha'(t) = \frac{f \circ \gamma \circ \alpha(t)(F^{-1}(\gamma, \gamma') \circ \alpha(t))_3^q (q+1) \parallel \tilde{F} \circ L_{\pm}^{-1} \circ F^{-1}(\gamma, \gamma') \circ \alpha(t) \parallel}{\lambda_* t(\|\tilde{F} \circ L_{\pm}^{-1} \circ F^{-1}(\gamma, \gamma') \circ \alpha(t) \parallel -1)}. \tag{2.3$_{+,-}$}$$

The theorem we have been aiming to prove is:

Theorem 2.4. *For* $b \in]-\frac{1}{9}, 0[$ *there exists a geodesic* $\gamma :]0, \epsilon[\to M$ *with*

$$\gamma'(t) \in \mathbb{D}(\phi_*^+), \quad t \in]0, \epsilon[$$

such that $(2.3)_+$ *has an increasing solution*

$$\alpha :]0, \delta[\to]0, \epsilon[, \quad \lim_{t \to 0_+} \alpha(t) = 0$$

making $\gamma' \circ \alpha$ + *resolvent semi-analytic*

$$\phi_*^+ \circ \gamma' \circ \alpha(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} \gamma^{\mathbf{p}} t^{\mathbf{pa}},$$

$$\gamma^{e_1} \in V^{\phi}(\tfrac{1}{2}\lambda y_2)^+, \quad \gamma^{e_2} \in V^{\phi}((q_+ - 1)\lambda y_2), \quad \gamma^{e_3} \in V^{\phi}(\tfrac{1}{2}(q_+ - 3)\lambda y_2).$$

For $b \in]\frac{3}{8}, +\infty[$ *there exists a geodesic* $\gamma :]0, \epsilon[\to M$ *with*

$$\gamma'(t) \in \mathbb{D}(\phi_*^+), \quad t \in]0, \epsilon[$$

such that $(2.3)_+$ *has an increasing solution*

$$\alpha :]0, \delta[\to]0, \epsilon[, \quad \lim_{t \to 0_+} \alpha(t) = 0$$

making $\gamma' \circ \alpha$ + *resolvent semi-analytic*

$$\phi_*^+ \circ \gamma' \circ \alpha(t) = \sum_{p_1=0}^{+\infty} \gamma^{p_1 e_1} t^{p_1}, \quad \gamma^{e_1} \in V^{\phi}(\tfrac{1}{2}\lambda y_2)^+.$$

For $b \in]-\frac{1}{8}, 1[$ *there exists a geodesic* $\gamma :]0, \epsilon[\to M$ *with*

$$\gamma'(t) \in \mathbb{D}(\phi_*^-), \quad t \in]0, \epsilon[$$

such that $(2.3)_-$ *has an increasing solution*

$$\alpha : 0, \delta[\to]0, \epsilon[, \quad \lim_{t \to 0_+} \alpha(t) = 0$$

making $\gamma' \circ \alpha$ - *resolvent semi-analytic*

$$\phi_*^- \circ \gamma' \circ \alpha(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \gamma^{\mathbf{p}} t^{\mathbf{pa}},$$

$$\gamma^{e_1} \in V^{\phi}\left(\frac{\lambda}{2} y_2\right)^+, \quad \gamma^{e_2} \in V^{\phi}((q_- - 1)\lambda y_2).$$

If $\sigma :]0, \epsilon_1[\to M$ *is a geodesic satisfying the above then* $\sigma = \gamma$ *in their common domain of definition.*

Proof. q_+ is increasing and

$$q_+(b) \to 3, \quad b \to -\tfrac{1}{8}, \qquad q_+(b) \to +\infty, \quad b \to 0_-,$$
$$q_+(b) \to -\infty, \quad b \to 0_+, \qquad q_+(b) \to 1, \quad b \to +\infty,$$

while q_- is decreasing and

$$q_-(b) \to 3, \quad b \to -\tfrac{1}{8}, \quad q_-(b) \to 1, \quad b \to 1.$$

It follows that for $b \in]-\frac{1}{8}, 0[$

$$(q_+ - 1)\lambda y_2 < 0, \quad \tfrac{1}{2}(q_+ - 3)\lambda y_2 < 0.$$

Now $q_+(b) = 2 - 4b/(4b + 1 - \sqrt{8b+1}) = 4$ is equivalent to $b = -\frac{1}{9}$.

So for $b \in]-\frac{1}{9}, 0[$ we have

$$q_+ - 1 > \tfrac{1}{2}, \quad \tfrac{1}{2}(q_+ - 3) > \tfrac{1}{2}.$$

Applying Lemma 3.4. there exists a semi-analytic curve

$$\beta(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} \gamma^{\mathbf{p}} t^{\mathbf{pa}},$$
$$\gamma^{e_1} \in V^{\phi}\left(\frac{\lambda}{2} y_2\right)^+, \quad \gamma^{e_2} \in V^{\phi}((q_+ - 1)\lambda y_2), \quad \gamma^{e_3} \in V^{\phi}(\tfrac{1}{2}(q_+ - 3)\lambda y_2),$$

such that

$$\beta'(t) = \frac{1}{\lambda_* t} \mathcal{Z}_*(\beta(t)),$$

β is C^1 and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{F}^{-1} \circ \beta)(0) = (0, y)\langle (0, y), \gamma^{e_1} \rangle, \qquad \tilde{F}^{-1} \circ \beta(0) = 0$$

So

$$\tilde{F}^{-1} \circ \beta(t) = tH(t)$$

for a continuous map H. Now $a_+ < 0$, so

$$f \circ F \circ L_+ \circ \tilde{F}^{-1} \circ \beta(t) = t^2 k(t), \quad k(0) < 0$$

for a continuous function k. Define

$$\tau(t) = \int_0^t \frac{f \circ F \circ L_+ \circ \tilde{F}^{-1} \circ \beta(s)(L_+ \circ \tilde{F}^{-1} \circ \beta(s))_3^q (q + 1) \parallel \beta(s) \parallel}{\lambda_* s(\parallel \beta(s) \parallel - 1)}$$

$$= \int_0^t s^q k_*(s)\,\mathrm{d}s$$

for a continuous function k_* having $k_*(0) > 0$.

It follows that $\tau :]0, \delta[\to]0, \epsilon[$ is increasing which means we can define

$$(\gamma, \gamma')(t) = F \circ L_+ \circ \tilde{F}^{-1} \circ \beta \circ \tau^{-1}(t).$$

If you differentiate this curve, you will find that it is an integral curve of the local representative of the geodesic spray. Furthermore τ solves $(2.3)_+$. γ is thus the local representative of a geodesic satisfying the requirements of the theorem.

The existence part of the theorem follows in this case.

To prove the uniqueness part of the theorem, define

$$\beta_*(t) = \phi_*^+ \circ \sigma' \circ \alpha(t)$$

and verify that

$$\beta_*'(t) = \frac{1}{\lambda_* t} \mathcal{Z}_*(\beta_*(t)).$$

According to Lemma 3.7. $\beta = \beta_*$ on their common domain of definition. Differentiate

$$(\alpha \circ \tau^{-1})'(t) = 1$$

using the definition of τ and $(2.3)_+$. It follows that $\tau = \alpha$ near 0. Now

$$\begin{aligned}(\sigma, \sigma')(t) &= F \circ L_+ \circ \tilde{F}^{-1}(\beta_* \circ \alpha^{-1})(t) \\ &= F \circ L_+ \circ \tilde{F}^{-1}(\beta \circ \tau^{-1})(t) = (\gamma, \gamma')(t)\end{aligned}$$

near 0 and the theorem follows in this case.

Now

$$q_+(b) = 2 - \frac{4b}{4b + 1 - \sqrt{8b+1}} = -1, \quad b \in]0, +\infty[,$$

is equivalent to $b = \frac{3}{8}$.

So for $b > \frac{3}{8}$

$$(q_+ - 1)\lambda y_2 > 0, \quad \tfrac{1}{2}(q_+ - 3)\lambda y_2 > 0.$$

Applying Lemma 3.4 there exists a semi-analytic curve

$$\beta(t) = \sum_{p_1=0}^{+\infty} \gamma^{p_1} t^{p_1}, \quad \gamma^1 \in V^\phi(\tfrac{1}{2}\lambda y_2)^+$$

such that

$$\beta'(t) = \frac{1}{\lambda_* t} \mathcal{Z}_*(\beta(t)).$$

Arguing as above the theorem follows in this case.

For $b \in]-\frac{1}{8}, +\infty[$

$$1 < q_-(b) < 3$$

hence

$$(q_- - 1)\lambda y_2 < 0, \quad \tfrac{1}{2}(q_- - 3)\lambda y_2 > 0.$$

Now $q_-(b) = 2 - 4b/(4b + 1 + \sqrt{8b+1}) = \frac{3}{2}$ is equivalent to $b = 1$.

So for $b \in]-\frac{1}{8}, 1[$ we have

$$q_- - 1 > \tfrac{1}{2}.$$

Applying Lemma 3.4 there exists a semi-analytic curve

$$\beta(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \gamma^{\mathbf{p}} t^{\mathbf{pa}},$$

$$\gamma^{e_1} \in V^{\Phi}(\tfrac{1}{2}\lambda y_2)^+, \quad \gamma^{e_2} \in V^{\Phi}((q_- - 1)\lambda y_2)$$

such that

$$\beta'(t) = \frac{1}{\lambda_* t} \mathcal{Z}_*(\beta(t)).$$

Arguments similar to the above leads to the conclusion of the theorem in this case. □

3. Semi-analytic curves

Let $A : U \to \mathbb{R}^n, \quad n = k + m$ be a real analytic vector field on an open neighbourhood U of 0 in $\mathbb{R}^n$ such that $A(0) = 0$ and

$$L = DA_0 = \begin{pmatrix} \lambda_1 & . & . & 0 \\ . & . & . & . \\ . & . & . & . \\ 0 & . & . & \lambda_n \end{pmatrix},$$

$$\lambda_i < 0 \;\; i = 1, \ldots, k, \lambda_i \geq 0 \;\; i = k+1, \ldots, n, \quad A = L + B.$$

We can assume that $\lambda_k \leq \lambda_{k-1} \leq \cdots \leq \lambda_1 = \lambda_*$ and define

$$\alpha_i = \lambda_i/\lambda_1, \quad \mathbf{a} = (\alpha_1, \ldots, \alpha_k), \quad \mathbf{p} = (p_1, \ldots, p_k), \quad p_i \in \mathbb{N}_0,$$

$$a = \mathbf{pa}.$$

We shall assume that

$$\mathbf{pa} \neq \alpha_i, \quad i = 1, \ldots, k, \qquad \mathbf{p} \neq e_1, \ldots, e_k, \tag{3.1}$$

where $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$. Define

$$\gamma(t) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \gamma^{\mathbf{p}} t^{\mathbf{pa}},$$
$$\gamma_i^{e_i} = v_i \in \mathbb{R},$$
$$\gamma_j^{e_i} = 0, \quad j \neq i, \quad j \in \{1, \ldots, n\}, \quad \gamma^0 = 0.$$

Assume $\gamma^{\mathbf{p}}$ is defined for $|\mathbf{p}| \leq l, l \geq 1$. Let $\mathbf{p}$ have $|\mathbf{p}| = l + 1$ and define

$$F^{\mathbf{p}} = \sum_{a=2}^{l+1} \sum_{\mathbf{p}_1 + \cdots + \mathbf{p}_a = \mathbf{p}} B_{i_1 \cdots i_a} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a}.$$

According to (3.1) we can solve

$$(\alpha_i / a - 1) \gamma_i^{\mathbf{p}} = -\frac{1}{\lambda_* a} F_i^{\mathbf{p}}$$

for $\gamma_j^{\mathbf{p}}$ which is then defined.

Definition 3.1. The curve

$$\tilde{\gamma}(t) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \tilde{\gamma}^{\mathbf{p}} t^{\mathbf{pa}}$$

is semi-analytic provided

$$|\tilde{\gamma}^{\mathbf{p}}| \leq \frac{C}{(p_1+1)^2 \cdots (p_k+1)^2 r^{p_1 + \cdots p_k}} \tag{3.2}$$

for some $C, r > 0$.

We can suppose that $r \in]0, 1[$.

Lemma 3.2. *$\tilde{\gamma}$ is well-defined for $t \in]0, \delta[, \ \delta \in]0, r[$.*

Proof. We have

$$\begin{aligned} \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} |\gamma^{\mathbf{p}} t^{\mathbf{pa}}| &\leq \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \frac{C|t|^{\mathbf{pa}}}{r^{p_1 + \cdots + p_k}} \\ &\leq \sum_{p_1=0}^{+\infty} \cdots \frac{C|t|^{p_1\alpha_1 + \cdots + p_{k-1}\alpha_{k-1}}}{r^{p_1 + \cdots + p_{k-1}}} \sum_{p_k=0}^{+\infty} \left(\frac{|t|^{\alpha_k}}{r}\right)^{p_k} \\ &\leq \frac{C}{(1 - |t|^{\alpha_1}/r) \cdots (1 - |t|^{\alpha_k}/r)}. \end{aligned}$$ □

It is going to be important for us to prove the following.

Lemma 3.3. *γ is semi-analytic.*

Proof. We have

$$\sum_{j_1+j_2=j,\, j_1,\, j_2\geq 0} \frac{1}{(j_1+1)^2(j_2+1)^2} \leq K_2/(j+1)^2$$

for some $K_2 > 0$ and all $j \geq 0$. Also

$$|B_{i_1\cdots i_a}| \leq D/\rho^a, \quad D, \rho > 0.$$

So

$$\begin{aligned}
|F^{\mathbf{p}}| &\leq \sum_{a=2}^{|\mathbf{p}|} \sum_{\mathbf{p}_1+\cdots+\mathbf{p}_a=\mathbf{p}} |B_{i_1\cdots i_a}||\gamma_{i_1}^{\mathbf{p}_1}|\cdots|\gamma_{i_a}^{\mathbf{p}_a}| \\
&\leq \sum_{a=2}^{|\mathbf{p}|} \sum_{p_1^1+\cdots+p_1^a=p_1} \cdots \sum_{p_k^1+\cdots+p_k^a=p_k} \frac{D}{\rho^a} \frac{C^a}{(p_1^1+1)^2\cdots(p_k^a+1)^2 r^{p_1+\cdots+p_k}} \\
&\leq \sum_{a=2}^{|\mathbf{p}|} \frac{D}{\rho^a} C^a \frac{1}{r^{p_1+\cdots+p_k}} \frac{K_2^{a-1}}{(p_1+1)^2} \cdots \frac{K_2^{a-1}}{(p_k+1)^2} \\
&= \frac{D}{K_2^k} \frac{1}{(p_1+1)^2\cdots(p_k+1)^2 r^{p_1+\cdots+p_k}} \sum_{a=2}^{|\mathbf{p}|} \left(\frac{CK_2^k}{\rho}\right)^a \\
&\leq \frac{2D}{K_2^k}(CK_2^k/\rho)^2 \frac{1}{(p_1+1)^2\cdots(p_k+1)^2 r^{p_1+\cdots+p_k}},
\end{aligned}$$

where we have arranged that

$$CK_2^k/\rho < \tfrac{1}{2}$$

choosing $C > 0$ appropriately. We can assume in addition that

$$2DCK_2^k/\rho^2 < |\lambda_*(1-\alpha_i/a)| \quad \forall \mathbf{p}, \quad |\mathbf{p}| = p_1 + \cdots + p_k \geq 1.$$

Now take $r > 0$ small enough that

$$|v_i| \leq C/4r.$$

Then

$$|\gamma_i^{\mathbf{p}}| = \frac{1}{|\lambda_* a(1-\alpha_i/a)|}|F_i^{\mathbf{p}}| \leq \frac{C}{(p_1+1)^2\cdots(p_k+1)^2 r^{p_1+\cdots+p_k}}.$$

The lemma follows. □

Suppose we have a sequence

$$a_{\mathbf{p}}, \quad \mathbf{p} \in \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$$

such that

$$\sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} |a_{\mathbf{p}}| < +\infty$$

then

$$\sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} a_{\mathbf{p}} = \sum_{m=0}^{+\infty} \sum_{p_1+\ldots+p_k=m} a_{\mathbf{p}}.$$

This is going to be important in proving the following lamma.

Lemma 3.4. $\gamma'(t) = (1/\lambda_* t)A(\gamma(t))$.

Proof. We have

$$A = L + B, \quad B(x) = \sum_{a=2}^{+\infty} B_{i_1\cdots i_a} x_{i_1} \cdots x_{i_a}.$$

We find

$$\begin{aligned}
B(\gamma(s)) &= \sum_{a=2}^{+\infty} B_{i_1\cdots i_a} \sum_{p_1^1=0}^{+\infty} \cdots \sum_{p_k^1=0}^{+\infty} \gamma_{i_1}^{\mathbf{p}_1} t^{\mathbf{p}_1\mathbf{a}} \cdots \sum_{p_1^a}^{+\infty} \cdots \sum_{p_k^a=0}^{=\infty} \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{p}_a\mathbf{a}} \\
&= \sum_{a=2}^{+\infty} B_{i_1\cdots i_a} \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \sum_{p_1^1+\cdots+p_1^a=p_1} \cdots \sum_{p_k^1+\cdots+p_k^a=p_k} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{pa}} \\
&= \sum_{a=2}^{+\infty} B_{i_1\cdots i_a} \sum_{m=a}^{+\infty} \sum_{p_1^1+\cdots+p_k^a=m} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{pa}} \\
&= \sum_{m=2}^{+\infty} \sum_{a=2}^{m} B_{i_1\cdots i_a} \sum_{p_1^1+\cdots+p_k^a=m} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{pa}} \\
&= \sum_{m=2}^{+\infty} \sum_{a=2}^{m} B_{i_1\cdots i_a} \sum_{p_1+\cdots+p_k=m} \sum_{\mathbf{p}_1+\cdots+\mathbf{p}_a=\mathbf{p}} \gamma_{i_1}^{\mathbf{p}_1} \cdots \gamma_{i_a}^{\mathbf{p}_a} t^{\mathbf{pa}} \\
&= \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} F^{\mathbf{p}} t^{\mathbf{pa}}.
\end{aligned}$$

Now we find

$$\begin{aligned}
&\int_0^t \frac{1}{\lambda_* s} A(\gamma(s))_i \, \mathrm{d}s \\
&= \int_0^t \frac{1}{s}(\alpha_i \gamma_i(s) + \frac{1}{\lambda_*} B_i(\gamma(s)))\mathrm{d}s \\
&= \int_0^t \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \left(\alpha_i \gamma_i^{\mathbf{p}} s^{\mathbf{pa}-1} + \frac{1}{\lambda_*} F_i^{\mathbf{p}} s^{\mathbf{pa}-1} \right) \mathrm{d}s
\end{aligned}$$

$$= \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \left(\alpha_i \gamma_i^{\mathbf{p}} \frac{1}{a} + \frac{1}{\lambda_* a} F_i^{\mathbf{p}} \right) t^{\mathbf{pa}}$$

$$= \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} \gamma_i^{\mathbf{p}} t^{\mathbf{pa}} = \gamma_i(t).$$

Now define

$$\beta(\mathbf{p}) = \mathbf{pa}$$
$$\{\beta(\mathbf{p}) \mid \mathbf{p} \in \mathbb{N}_0 \times \cdots \times \mathbb{N}_0\} = \{\beta_1, \beta_2, \ldots\} \quad \beta_1 < \beta_2 < \cdots$$

Also let

$$\gamma^i = \sum_{\mathbf{pa}=\beta_i} \gamma^{\mathbf{p}}.$$

Then we have:

Lemma 3.5. $\gamma(t) = \sum_{i=1}^{+\infty} \gamma^i t^{\beta_i}, \quad t \in]0, \delta[.$

Suppose we have a curve

$$\tilde{\gamma}(t) = \sum_{i=1}^{+\infty} \tilde{\gamma}^i t^{\beta_i}, \quad \tilde{\gamma}^i = \sum_{\mathbf{pa}=\beta_i} \tilde{\gamma}^{\mathbf{p}},$$

where the $\tilde{\gamma}^{\mathbf{p}}$ satisfies (3.2).

Lemma 3.6. *If* $\tilde{\gamma}(t) = 0,\ t \in]0, \delta[$ *then* $\tilde{\gamma}^i = 0 \forall i \in \mathbb{N}$.

Proof. To see that $\tilde{\gamma}^1 = 0$ compute for $t \in]0, \delta[$

$$\frac{1}{t^{\beta_1}} \tilde{\gamma}(t) = \frac{1}{t^{\beta_1}} \left\{ \tilde{\gamma}^1 t^{\beta_1} + \sum_{i=2}^{+\infty} \tilde{\gamma}^i t^{\beta_i} \right\}$$
$$= \tilde{\gamma}^1 + t^{\beta_2 - \beta_1} \sum_{i=2}^{+\infty} \tilde{\gamma}^i t^{\beta_i - \beta_2}$$
$$\to \tilde{\gamma}^1 = 0$$

as $t \to 0$. Assume $\tilde{\gamma}^1 = \cdots = \tilde{\gamma}^N = 0$.

Then

$$\frac{1}{t^{\beta_{N+1}}} \tilde{\gamma}(t) = \tilde{\gamma}^{N+1} + t^{\beta_{N+2} - \beta_{N+1}} \sum_{i=N+2}^{+\infty} \tilde{\gamma}^i t^{\beta_i - \beta_{N+1}} \to \tilde{\gamma}^{N+1} = 0$$

as $t \to 0$. The lemma follows. □

We are now in a position to prove the following result.

Lemma 3.7. *If*

$$\gamma(t) = \sum_{i=1}^{+\infty} \gamma^i t^{\beta_i}, \quad \tilde{\gamma}(t) = \sum_{i=1}^{+\infty} \tilde{\gamma}^i t^{\beta_i}, \quad t \in]0, \delta[$$

are two semi-analytic curves satisfying

$$\gamma'(t) = \frac{1}{\lambda_* t} A(\gamma(t)), \quad \tilde{\gamma}'(t) = \frac{1}{\lambda_* t} A(\tilde{\gamma}(t)), \quad t \in]0, \delta[$$
$$\gamma^{e_1} = \tilde{\gamma}^{e_1} = v_1 e_1$$
$$\vdots$$
$$\gamma^{e_k} = \tilde{\gamma}^{e_k} = v_k e_k.$$

Then $\gamma = \tilde{\gamma}$.

Proof. We have

$$F^k = \sum_{\mathbf{pa}=\beta_k} F^{\mathbf{p}} = \sum_{a=2}^{k} \sum_{\beta_{j_1}+\cdots+\beta_{j_a}=\beta_k} B_{i_1\cdots i_a} \gamma_{i_1}^{j_1} \cdots \gamma_{i_a}^{j_a}.$$

Hence

$$B(\gamma(t)) = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_k=0}^{+\infty} F^{\mathbf{p}} t^{\mathbf{pa}} = \sum_{i=1}^{+\infty} \left(\sum_{\mathbf{pa}=\beta_i} F^{\mathbf{p}} \right) t^{\beta_i} = \sum_{i=1}^{+\infty} F^i t^{\beta_i},$$

we obtain

$$\sum_{i=1}^{+\infty} \left(\alpha_j \gamma_j^i \frac{1}{\beta_i} + \frac{1}{\lambda_* \beta_i} F_j^i \right) t^{\beta_i} = \sum_{i=1}^{+\infty} \gamma_j^i t^{\beta_i}.$$

By Lemma 3.6

$$\alpha_j \gamma_j^i \frac{1}{\beta_i} + \frac{1}{\lambda_* \beta_i} F_j^i = \gamma_j^i \quad \forall i \in \mathbb{N}.$$

Now

$$\gamma^1 = \gamma^{e_1} = \tilde{\gamma}^{e_1} = \tilde{\gamma}^1.$$

Assume

$$\gamma^i = \tilde{\gamma}^i, \quad i = 1, \ldots, N.$$

If $\beta_{N+1} = \alpha_j$ for some $j \in \{1, \ldots, k\}$ then

$$\gamma^{N+1} = \gamma^{e_j} = \tilde{\gamma}^{e_j} = \tilde{\gamma}^{N+1}.$$

Otherwise

$$\begin{aligned}\gamma_j^{N+1} &= \frac{1}{\lambda_* \beta_{N+1}(1-\alpha_j/\beta_{N+1})} F_j^{N+1} \\ &= \frac{1}{\lambda_* \beta_{N+1}(1-\alpha_j/\beta_{N+1})} \tilde{F}_j^{N+1} \\ &= \tilde{\gamma}_j^{N+1}\end{aligned}$$

and the lemma follows. □

4. Three collision orbits in the Helium atom

This section is devoted to the study of the orbit structure of the Hamiltonian vector field on

$$M = T^*Q, \quad Q = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 > 0\}$$

with the standard symplectic form

$$\omega = \sum_{i=1}^{2} \mathrm{d}x_i \wedge \mathrm{d}p_i$$

and Hamiltonian

$$H(x, p) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 - 2/x_1 - 2/x_2 + 1/(x_1 + x_2).$$

This is the Hamiltonian of the Helium atom and gives rise to the Hamiltonian vector field

$$X_H(x, p) = \begin{pmatrix} p_1 \\ p_2 \\ -2/x_1^2 + 1/(x_1 + x_2)^2 \\ -2/x_2^2 + 1/(x_1 + x_2)^2 \end{pmatrix}.$$

We are going to be interested in orbits $\gamma = (\gamma_1, \gamma_2) :]0, \epsilon[\to M$ of X_H with the property that $\gamma_1(t) \to 0$ as $t \to 0$. This corresponds after time reversal to the simultaneous collision of the two electrons with the nucleus. Such an orbit is called a three-collision orbit. Let L denote the linear map with matrix representation

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

in the standard basis $\{e_i\}$ of $\mathbb{R}^4$ and define

$$h(x) = (x_1 + x_2)^2 (x_1 + x_2)^2 x_1^2.$$

We then get a new vector field

$$Y(x,p) = L^{-1} \circ X_H \circ L(x,p) \cdot h(x)$$
$$= \begin{pmatrix} p_1 h(x) \\ p_2 h(x) \\ \frac{1}{2}(-2(x_1-x_2)^2 x_1^2 - 2(x_1+x_2)^2 x_1^2 + \frac{1}{2}(x_1+x_2)^2(x_1-x_2)^2) \\ 4x_1^3 x_2 \end{pmatrix}.$$

Recall the definition of G and Φ from Section 2 now letting $q = \frac{1}{2}$. They gave rise to the map

$$F(v,p) = (G(v), \Phi(p)).$$

Now define

$$Z(v,p) = v_1 p_1^{1/2} DF^{-1}(Y(F(v,p)))$$
$$= Z_7(v,p) + \text{higher order terms}.$$

Here

$$Z_7(v,p) = \begin{pmatrix} v_1^7 \\ (-v_2+p_2)v_1^6 \\ \frac{7}{2}p_1^2 v_1^5 \\ \frac{7}{4}p_1 p_2 v_1^5 + 4p_1 v_2 v_1^5 \end{pmatrix},$$

Z is the restriction of a real analytic vector field on $\mathbb{R}^4$ also denoted Z. Now define

$$\mathcal{Z}(x) = \frac{\| x \|^6}{(\| x \| - 1)^6} D\tilde{F}(Z(\tilde{F}^{-1}(x))), \quad \| x \| > 1,$$
$$\mathcal{Z}(x) = \frac{\| x \|^6}{(\| x \| - 1)^6} D\tilde{G}(Z(\tilde{G}^{-1}(x))), \quad 0 < \| x \| < 1,$$

which is the restriction of a real analytic vector field also denoted $\mathcal{Z}$ on $\mathbb{R}^4 \backslash \{0\}$. So is

$$\mathcal{Z}_7(x) = \frac{\| x \|^6}{(\| x \| - 1)^6} D\tilde{F}(Z_7(\tilde{F}^{-1}(x)))$$
$$= Z_7(x) - x / \| x \|^3 \langle x, Z_7(x) \rangle.$$

Notice that

$$(v,p) = x_* = (7/\sqrt{53}, 0, 2/\sqrt{53}, 0)$$

is a singular point of $\mathcal{Z}$ and $\mathcal{Z}_7$ because

$$\mathcal{Z}_7(x_*) = (v_1^8 + \tfrac{7}{2}p_1^3 v_1^5)x_* = \lambda x_*.$$

Now

$$D\mathcal{Z}_7(x_*)$$
$$= \begin{pmatrix} 7v_1^6 - (9v_1^8 + 21p_1^3 v_1^5) + 3v_1^2\lambda & 0 & -\frac{21}{2}p_1^2 v_1^6 + 3v_1 p_1 \lambda & 0 \\ 0 & -v_1^6 - \lambda & 0 & v_1^6 \\ \frac{35}{2}p_1^2 v_1^4 - (8v_1^7 p_1 + \frac{35}{2}p_1^4 v_1^4) + 3v_1 p_1 \lambda & 0 & 7p_1 v_1^5 - (v_1^8 + 14p_1^3 v_1^5) + 3p_1^2 \lambda & 0 \\ 0 & 4p_1 v_1^5 & 0 & \frac{7}{4}p_1 v_1^5 - \lambda \end{pmatrix}$$
$$= \{\mathcal{Z}_{ij}\}.$$

The spectrum of this matrix is important in order to determine the three collision orbits.

We first determine the spectrum of

$$\begin{pmatrix} -v_1^6 - \lambda & v_1^6 \\ 4p_1 v_1^5 & \frac{7}{4} p_1 v_1^5 - \lambda \end{pmatrix}$$
$$= \begin{pmatrix} -v_1^6(v_1^2 + p_1^2) - \lambda & v_1^6(v_1^2 + p_1^2) \\ 4p_1 v_1^5(v_1^2 + p_1^2) & \frac{7}{4} p_1 v_1^5(v_1^2 + p_1^2) - \lambda \end{pmatrix}$$
$$= v_1^8 \frac{53}{49 \cdot 14} \begin{pmatrix} -28 & 14 \\ 16 & -7 \end{pmatrix}.$$

We have used that $p_1 = \frac{2}{7} v_1$. The determinant of this matrix is negative so there is one negative eigenvalue and one positive. We are only interested in the positive eigenvalue which is

$$\lambda_* = v_1^8 \frac{53}{49 \cdot 28}(-35 + \sqrt{1337}) = 0,04417.$$

This is a simple eigenvalue, the eigenspace being spanned by

$$\begin{pmatrix} 0 \\ 14 \\ 0 \\ \frac{1}{2}(21 + \sqrt{1337}) \end{pmatrix},$$

λ is an eigenvalue of geometric and algebraic multiplicity 2 for the matrix

$$A = \begin{pmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{13} \\ \mathcal{Z}_{31} & \mathcal{Z}_{33} \end{pmatrix}.$$

To see this compute

$$A\left(\begin{pmatrix} v_1 \\ p_1 \end{pmatrix}\right) = \begin{pmatrix} v_1(v_1^8 - 21 p_1^3 v_1^5 + 7 v_1^6 p_1^2) \\ p_1(v_1^8 + \frac{21}{2} p_1^3 v_1^5 - 7 p_1^3 v_1^5) \end{pmatrix}$$
$$= (v_1^8 + \tfrac{7}{2} v_1^5 p_1^3) \begin{pmatrix} v_1 \\ p_1 \end{pmatrix}$$

and

$$A\left(\begin{pmatrix} -p_1 \\ v_1 \end{pmatrix}\right) = \begin{pmatrix} -p_1(v_1^8 + \frac{49}{2} p_1^3 v_1^5 - 21 p_1^3 v_1^5) \\ v_1(v_1^8 + (\frac{56}{2} - \frac{35}{2} - 7) p_1^3 v_1^5) \end{pmatrix}.$$
$$= (v_1^8 + \tfrac{7}{2} p_1^3 v_1^5) \begin{pmatrix} -p_1 \\ v_1 \end{pmatrix}.$$

Notice that

$$Z_8(x) = \frac{1}{8!} D^8 Z_0(x, \ldots, x) = 0.$$

So

$$D\mathcal{Z}_7(x_*) = D\mathcal{Z}(x_*).$$

This means that

$$w_2 = \begin{pmatrix} v_1 \\ 0 \\ p_1 \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} -p_1 \\ 0 \\ v_1 \\ 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 \\ 14 \\ 0 \\ \frac{1}{2}(21+\sqrt{1337}) \end{pmatrix}$$

are eigenvectors of $D\mathcal{Z}(x_*)$ corresponding to eigenvalues

$$v_1^8 + \frac{7}{2} p_1^3 v_1^5 = v_1^8 \frac{53}{49}, \quad v_1^8 \frac{53}{49}, \quad v_1^8 \frac{53}{49 \cdot 28}(-35+\sqrt{1337}).$$

Define α_3 by

$$\alpha_3 \, v_1^8 \frac{53}{49 \cdot 28}(-35+\sqrt{1337}) = v_1^8 \frac{53}{49},$$

which implies

$$P(\alpha_3) = -112\alpha_3^2 + 70 \cdot 28\alpha_3 + 28^2 = 0.$$

Now

$$P(0) = 28^2 > 0, \quad P(17) = 1736 > 0, \quad P(18) = -224 < 0.$$

We conclude that $17 < \alpha_3 < 18$.

Define $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3) = (1, \alpha_3, \alpha_3)$.

Theorem 4.1. *For every $b_1 > 0, b_2 \in \mathbb{R}$ there exists a semi-analytic curve*

$$\gamma_1(t) = \left(\sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \gamma_1^{\mathbf{p}} t^{\mathbf{pa}}, \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \gamma_2^{\mathbf{p}} t^{\mathbf{pa}} \right), \quad t \in]0, \epsilon[,$$
$$\gamma_1^{pe_1} = 0, \quad p \in \mathbb{N}_0, \quad \gamma_1^{e_2} = b_1,$$
$$\gamma_2^{pe_1} = \gamma_2^{pe_1+e_2} = \gamma_2^{pe_1+e_3} = 0, \quad p \in \mathbb{N}_0,$$
$$\gamma_2^{e_1+2e_2} = b_2, \quad \gamma_2^{e_1+2e_3} = \gamma_2^{e_1+e_2+e_3} = \gamma_2^{2e_2} = \gamma_2^{2e_3} = \gamma_2^{e_2+e_3} = 0$$

such that $L \circ \gamma = L \circ (\gamma_1, \gamma_2) :]0, \epsilon[\to \mathbb{R}^4$ can be reparametrized to an integral curve of X_H. The reparametrization function is the inverse of a semi-analytic function

$$\tau(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \tau^{\mathbf{p}} t^{\mathbf{pa}_*}, \quad t \in]0, \delta[, \quad a_* = (1, \tfrac{1}{2}\alpha_3)$$
$$\tau^{\mathbf{p}} = 0, \quad \mathbf{pa}_* < \tfrac{3}{2}\alpha_3, \quad \sum_{\mathbf{pa}_* = \frac{3}{2}\alpha_3} \tau^{\mathbf{p}} > 0$$

Proof. Define

$$\beta(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} \beta^{\mathbf{p}} t^{\mathbf{pa}}, \quad t \in]0, \epsilon[$$

with

$$\beta^{e_i} \in \text{span}\,\{w_i\}, \quad \beta^0 = x_*$$

such that

$$\beta'(t) = \frac{1}{\lambda_* t} \mathcal{Z}(\beta(t)).$$

This is possible by Section 3.

Also define

$$k(x) = \frac{\| x \| - 1}{\| x \|}, \quad x \in \mathbb{R}^4 \backslash \{0\}.$$

As the composition of a real analytic function with a semi-analytic curve

$$h(t) = k \circ \beta(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} k^{\mathbf{p}} t^{\mathbf{pa}}$$

is a semi-analytic curve.

$$h(0) = k(\beta(o)) = k^0 = 0.$$

Let $\phi : U \to S^3$ denote a coordinate system on S^3 around $\phi(0) = x_*$ and define

$$\psi(r, x) = r\phi(x), \quad x \in U,$$

ψ is then a coordinate system on $\mathbb{R}^4$ around x_* for (r, x) near $(1, 0)$. We can assume that

$$\frac{\partial}{\partial x_1}(x_*) = w_1 / \| w_1 \|, \quad \frac{\partial}{\partial x_2}(x_*) = w_3 / \| w_3 \|, \quad \frac{\partial}{\partial r}(x_*) = w_2 / \| w_2 \|,$$

$\partial / \partial x_3(x_*)$ being an eigenvector corresponding to the negative eigenvalue of $D\mathcal{Z}(x_*)$. Notice that

$$\mathcal{Z}_1^{\psi}(1, x) = 0,$$

hence

$$\frac{\partial^s \mathcal{Z}_1^{\psi}}{\partial x_1^s}(1, 0) = 0.$$

Take

$$\beta_*(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} \beta_*^{\mathbf{p}} t^{\mathbf{pa}}$$

such that

$$\beta_*'(t) = \frac{1}{\lambda_* t} \mathcal{Z}^{\psi}(\beta_*(t))$$

and

$$D\psi(\beta_*^{e_i}) = \beta^{e_i}, \quad \beta_*^0 = e_1.$$

Then

$$\beta_*^{se_1} = 0, \quad s \geq 2.$$

By the uniqueness of β

$$\psi \circ \beta_* = \beta.$$

Now

$$h(t) = \frac{\| \beta(t) \| - 1}{\| \beta(t) \|} = \frac{\beta_*^1(t) - 1}{\beta_*^1(t)}.$$

We find

$$k^{re_1} = 0, \; r \in \mathbb{N}_0, \quad k^{e_2} = \beta_*^{e_2}{}_1 = a_1, \quad k^{e_3} = 0.$$

We choose $a_1 > 0$. Define

$$f_i^{\mathbf{r}} = \sum_{p_1+q_1=r_1} \sum_{p_2+q_2=r_2} \sum_{p_3+q_3=r_3} \beta_i^{\mathbf{p}} k^q$$

and verify that

$$\beta_i(t)h(t) = \sum_{r_1=0}^{+\infty} \sum_{r_2=0}^{+\infty} \sum_{r_3=0}^{+\infty} f^{\mathbf{r}} t^{\mathbf{ra}}.$$

Then

$$f^{re_1} = 0, \; r \in \mathbb{N}_0, \quad f_1^{e_2} = v_1 a_1, \quad f_1^{e_3} = 0$$

and

$$f_2^{e_2} = 0, \; f_2^{e_3} = 0, \; f_2^{e_1+e_2} = \beta_2^{e_1} k^{e_2} = a_2 a_1, \; f_2^{e_1+e_3} = 0.$$

Now

$$G(\beta_1(t)h(t), \beta_2(t)h(t)) = \sum_{r_1=0}^{+\infty} \sum_{r_2=0}^{+\infty} \sum_{r_3=0}^{+\infty} \gamma^{\mathbf{r}} t^{\mathbf{ra}} = \gamma_1(t).$$

Now use the above to verify that

$$\gamma_2^{re_1} = 0, \quad \gamma_2^{re_1+e_2} = 0, \quad \gamma_2^{re_1+e_3} = 0, \quad r \in \mathbb{N}_0$$

and

$$\gamma_2^{e_1+2e_3} = \gamma_2^{e_1+e_2+e_3} = \gamma_2^{2e_2} = \gamma_2^{e_2+e_3} = \gamma_2^{2e_3} = 0, \quad \gamma_2^{e_1+2e_2} = v_1 a_1^2 a_2.$$

Define

$$\tau(t) = \int_0^t \frac{(\tilde{F}^{-1} \circ \beta(s))_1 (\tilde{F}^{-1} \circ \beta(s))_3^{1/2} \parallel \beta(s) \parallel^6 h(F \circ \tilde{F}^{-1} \circ \beta(s))}{\lambda_* s(\parallel \beta(s) \parallel -1)^6} \, ds$$

$$= \int_0^t s^{3\alpha_3/2-1} l(s)\, ds, \quad t \in]0, \delta[$$

for a semi-analytic function l and verify that

$$L \circ F \circ \tilde{F}^{-1} \circ \beta \circ \tau^{-1}(t) = L \circ \gamma \circ \tau^{-1}(t)$$

is an integral curve of X_H thereby proving the theorem. □

Remark 4.2. So for any $b_1 > 0$, $b_2 \in \mathbb{R}$ there exists a curve

$$\gamma_1(t) = (b_1 t^{\alpha_3}, b_2 t^{2\alpha_3+1}) + \text{higher order terms}$$

such that $L \circ \gamma = L \circ (\gamma_1, \gamma_2)$ can be reparametrized to an integral curve of X_H.
The reparametrization function being the inverse of

$$\tau(t) = a t^{3\alpha_3/2} + \text{higher order terms}, \quad a > 0.$$

Continuing in the notation of the proof of Theorem 4.1, notice that

$$\mathcal{Z}^i(v_1, 0, p_1, 0) = 0, \quad i = 2, 4.$$

So we can define a new vector field

$$Y(v_1, p_1) = (\mathcal{Z}^1(v_1, 0, p_1, 0), \mathcal{Z}^3(v_1, 0, p_1, 0))$$

(x_*^1, x_*^3) is a singular point of Y and

$$DY(x_*^1, x_*^3) = A,$$

which has λ as double eigenvalue. Let

$$\xi(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \xi^{\mathbf{p}} t^{p_1+p_2},$$

$$\xi^0 = (x_*^1, x_*^3), \quad \xi^{e_1} = k(x_*^1, x_*^3), \quad k > 0, \quad \xi^{e_2} \in \text{span}(-x_*^3, x_*^1)$$

be chosen to satisfy

$$\xi'(t) = \frac{1}{\lambda t} Y(\xi(t)),$$

which is possible by Section 3. Define

$$\beta_*(t) = (\xi_1(t), 0, \xi_2(t), 0).$$

Then

$$\beta_*'(t) = \frac{1}{\lambda t} \mathcal{Z}(\beta_*(t)).$$

Letting

$$\beta(t) = \beta_*(t^{\alpha_3})$$

we find

$$\beta'(t) = \frac{1}{\lambda_* t} \mathcal{Z}(\beta(t))$$

with

$$\beta(t) = \sum_{p_1=0}^{+\infty} \sum_{p_2=0}^{+\infty} \sum_{p_3=0}^{+\infty} \beta^{\mathbf{p}} t^{\mathbf{pa}},$$
$$\beta^{e_1} = 0, \quad \beta^{e_i} \in \operatorname{span}\{w_i\}, \quad i = 2, 3.$$

Notice that

$$\beta^{\mathbf{p}} = 0, \quad p_1 \neq 0.$$

Define

$$(x, x, p, p)(t) = L \circ F \circ \tilde{F}^{-1} \circ \beta \circ \tau^{-1}(t).$$

We see that choosing $\beta^{e_1} = 0$ gives an integral curve (x_1, x_2, p_1, p_2) of X_H with

$$x_1 = x_2, \quad p_1 = p_2.$$

Reparamtetrizing β_* with

$$\eta(t) = \int_0^t \frac{(\tilde{F}^{-1} \circ \beta_*(s))_1 (\tilde{F}^{-1} \circ \beta_*(s))_3^{1/2} \parallel \beta_*(s) \parallel^6 h(F \circ \tilde{F}^{-1} \circ \beta_*(s))}{\lambda s (\parallel \beta_*(s) \parallel -1)^6} \, \mathrm{d}s$$
$$= \int_0^t s^{1/2} h(s) \, \mathrm{d}s = s^{3/2} k(s)$$

for real analytic functions h, k with $h(0), k(0) > 0$ we also get an integral curve

$$(x, x, p, p)(t) = L \circ F \circ \tilde{F}^{-1} \circ \beta_* \circ \eta^{-1}(t).$$

Define $\tau_*(t) = t^{3/2}$ and substitute $s = \tau(v)$ in

$$\tau_*^{-1}(t) = \int_0^t \frac{2}{3} s^{-1/3} \, \mathrm{d}s = \int_0^{\eta^{-1}(t)} \frac{2}{3} \frac{h(v)}{k(v)^{1/3}} \, \mathrm{d}v.$$

Define a real analytic function

$$\mu(s) = \int_0^s \frac{2}{3} \frac{h(v)}{k(v)^{1/3}} \, \mathrm{d}v$$

with $\mu'(0) > 0$. So by the inverse function theorem μ^{-1} exists and is real analytic, leading to the conclusion that

$$\begin{aligned} x(t) &= (L \circ F \circ \tilde{F}^{-1} \circ \beta_* \circ \eta^{-1})_1(t) \\ &= (L \circ F \circ \tilde{F}^{-1} \circ \beta_* \circ \mu^{-1} \circ \tau_*^{-1})_1(t) \\ &= f(t^{2/3}) \end{aligned}$$

for a real analytic function f on a neighbourhood of the origin.

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